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Quantized equations of motion in non-commutative theories

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Abstract. Quantum field theories based on interactions which contain the Moyal star product suffer, in the general case when time does not commute with space, from several diseases: quantum equation of motions contain unusual terms, conserved currents cannot be defined and the residual spacetime symmetry is not maintained. All these problems have the same origin: time ordering does not commute with taking the star product. Here we show that these difficulties can be circumvented by a new definition of time ordering: namely with respect to a light-cone variable. In particular the original spacetime symmetries $SO(1,1) \times SO(2)$ and translation invariance turn out to be respected. Unitarity is guaranteed as well.

1 Introduction

Space and time will, at extremely short distances, require new notions in both mathematical description and physical content. A simple physical argument for this is based on the uncertainty principle which says that black holes can be formed, thus leading to a horizon and other consequences when precision in time is high enough [1]. As a modest step into this direction one may understand the introduction of Moyal products in otherwise rather conventional flat space-time quantum field theory. They arise when the coordinates are being considered as Hermitian operators which satisfy simple commutation relations like

$$[x_\mu, x_\nu] = \mathrm{i}\theta_{\mu\nu}. \tag{1}$$

A typical interaction then reads

$$S_{\mathrm{int}} = g \int \mathrm{d}^4x\, \phi(x) * \phi(x) * \phi(x). \tag{2}$$

The discussion for the case when time/space commutators vanish is fairly advanced, whereas the case when they do not vanish is not yet very well understood. Although Feynman rules have been proposed which lead to unitarity in non-gauge theories [2–5], gauge theories seem to be inconsistent [6,7]. A somewhat more detailed study also reveals that quantum equations of motion have a form which is intractable in practice [8], but worse is the fact that symmetries which are present on the classical level do not seem to be maintained after quantization. The main truly disturbing example is $SO(1,1) \times SO(2)$ invariance [8]. This is of course not tolerable: we wish to characterise theories by their symmetry content, hence every deviation from a classically realised symmetry must be very well understood until we accept it as unavoidable.

In the present paper we first recall the symmetry content on the classical level as being generically $SO(1,1) \times SO(2)$. Since the conventional time ordering is not in accordance with this symmetry and thus the reason for its breakdown, we define a new notion of time ordering and explore the consequences of this change. It turns out that the perturbation theory formulated on this basis has all desired properties: it is compatible with the symmetry, leads to simple Feynman rules and closed expressions for the quantum equations of motion. The LSZ asymptotic condition can be formulated and unitarity is maintained. All of this will be derived for scalar field theories as example and is still restricted to the tree approximation (apart from the unitarity relations which involve one-loop contributions). But the generality of the results supports the hope that proceeding in the direction of renormalisation and incorporating gauge theories will be possible.

2 Symmetries and standard form

We would like to show first that all non-commutative field theories defined from an action via the Moyal product are either $SO(1,1) \times SO(2)$ invariant, or have the symmetries of the so-called light-like case (the product of two null rotations). To see this consider for simplicity a non-commutative scalar field theory (for example scalar ϕ^3_* theory). The action $S[\phi;\theta]$, is a functional of the fields ϕ and a function of the constant θ matrix $\theta^{\mu\nu}$. Now given such an action with an arbitrary θ we change basis so that a point previously specified by coordinates

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x^μ is now specified by the coordinates $x'^\mu = L^\mu{}_\nu x^\nu$ where $L \in SO(1,3)$. Then the scalar field ϕ transforms to ϕ' defined as $\phi'(x) = \phi(L^{-1}x)$ and it follows that derivatives of the scalar field $\partial_\mu\phi$ transform to $L^\nu{}_\mu\partial_\nu\phi'$. One can then see that for a theory whose Lorentz violation comes only from the Moyal star we have

$$S[\phi;\theta] = S[\phi'; L\theta L^{\mathrm{T}}]. \tag{3}$$

So by a simple change of coordinates (i.e. we make no physical change) the transformed action has a similar form to the original action but with θ replaced by $L\theta L^{\mathrm{T}}$. This means that starting with any theory defined in terms of an arbitrary θ we may change coordinate basis in order that θ has the simplest form possible.

Let us note parenthetically that the situation has a strong analogy with a broken internal symmetry. The θ can be thought of as a field taking on a constant expectation value which then does not propagate. In this case the fields arrange themselves in representations of the larger symmetry although in fact only the symmetry which leaves the expectation value invariant is preserved. Similarly here we expect to be able to use the field representations of the full Lorentz symmetry even though this has been broken down to a smaller group.

In order to find the simplest form for θ it is easier to work with the spinorial representation of θ. We have

$$\theta^{\mu\nu} = \tau^{\mu\nu}_{\alpha\beta}\boldsymbol{\theta}^{\alpha\beta} + \overline{\tau}^{\mu\nu}_{\dot\alpha\dot\beta}\overline{\boldsymbol{\theta}}^{\dot\alpha\dot\beta}, \tag{4}$$

where $\alpha, \beta = (1,2)$ are Weyl spinor indices which transform under $SL(2;\mathbb{C}) \sim SO(1,3)$ and τ are the Pauli matrices. So $\theta^{\mu\nu}$ is equivalent to a complex symmetric 2×2 matrix $\boldsymbol{\theta}^{\alpha\beta}$ transforming as $\boldsymbol{\theta}' = M\boldsymbol{\theta}M^{\mathrm{T}}$ where $M \in SL(2;\mathbb{C})$ is a complex 2×2 matrix with unit determinant. It is easy to check that as long as the determinant of $\boldsymbol{\theta}$ is non-zero there exists an $M \in SL(2;\mathbb{C})$ such that

$$\theta' = M\theta M^{\mathrm{T}} = \sqrt{\det(\boldsymbol{\theta})} I_2. \tag{5}$$

The remaining symmetry which leaves θ' invariant is clearly $SO(2;\mathbb{C})$.

If, on the other hand, the determinant of $\boldsymbol{\theta}$ vanishes then either $\boldsymbol{\theta} = 0$ or there is an M such that

$$\theta' = M\theta M^{\mathrm{T}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

This corresponds to the "light-like" case of Aharony, Gomis and Mehen [9]. The remaining symmetry in this case is given by 2×2 matrices of the form

$$M = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad b \in \mathbb{C}. \tag{7}$$

If one then translates this back we find that $\theta^{\mu\nu}$ is always equivalent to one of the following forms:

$$\begin{pmatrix} 0 & \theta_e & 0 & 0 \\ -\theta_e & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_m \\ 0 & 0 & -\theta_m & 0 \end{pmatrix} \tag{8a}$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \tag{8b}$$

In the first case the remaining symmetry is $SO(1,1) \times SO(2)$ in general, extended to $O(1,1) \times SO(2)$ if $\theta_e = 0, \theta_m \neq 0$, to $SO(1,1) \times O(2)$ if $\theta_e \neq 0, \theta_m = 0$ and of course to the full $SO(1,3)$ if $\theta_e = \theta_m = 0$. In the latter case, the "light-like case" the remaining symmetry is harder to describe. It consists of two "null rotations" [10] both of which leave $x^0 - x^1$ invariant. The symmetry is given by $x^\mu \to L^\mu{}_\nu x^\nu$ with

$$L^\mu{}_\nu = \begin{pmatrix} 1+\frac{1}{2}(a^2+b^2) & -\frac{1}{2}(a^2+b^2) & a & b \\ \frac{1}{2}(a^2+b^2) & 1-\frac{1}{2}(a^2+b^2) & a & b \\ a & -a & 1 & 0 \\ b & -b & 0 & 1 \end{pmatrix}. \tag{9}$$

Since for *any* θ we can choose coordinates such that in the new coordinates θ takes one of the forms of equation (8), it follows that *any* theory defined by a non-zero θ is invariant under either $(S)O(1,1) \times (S)O(2)$ or the afore-mentioned symmetry of the light-like case. Of course in the original coordinates then these symmetries will, in general, be difficult to see.

Furthermore we see that in the space of non-commutative theories almost all cases can be given in terms of a θ of the form (8a) with $\theta_e \neq 0, \theta_m \neq 0$. This is thus the generic case and the case which we will concentrate on in the rest of this paper.

3 Locality properties and time ordering

3.1 Commutation relations

It is obvious from the definition of the Moyal product that the locality properties of the theory will drastically differ from those of an ordinary quantum field theory. To begin with let us consider commutators of composite operators in an ordinary free theory. We have real scalar fields $\Phi(x)$ which we split into positive and negative frequency parts as $\Phi(x) = \Phi^+(x) + \Phi^-(x)$ in the usual way. We canonically quantize the theory and define the commutator functions

$$\mathrm{i}\Delta^+(x-y) = \left[\Phi^+(x), \Phi^-(y)\right], \tag{10}$$

$$\begin{aligned} \mathrm{i}\Delta^-(x-y) &= \left[\Phi^-(x), \Phi^+(y)\right] \\ &= -\mathrm{i}\Delta^+(y-x), \end{aligned} \tag{11}$$

$$\begin{aligned} \mathrm{i}\Delta(x-y) &= [\Phi(x), \Phi(y)] \\ &= \mathrm{i}\Delta^+(x-y) + \mathrm{i}\Delta^-(x-y). \end{aligned} \tag{12}$$

Using standard identities of commutators one finds

$$\begin{aligned} \left[\Phi(x), \Phi^2(y)\right] &= 2[\Phi(x), \Phi(y)]\Phi(y) \\ &= 2\mathrm{i}\Delta(x-y)\Phi(y), \end{aligned} \tag{13}$$

$$\left[\Phi^2(x), \Phi^2(y)\right] = 2\mathrm{i}\Delta(x-y)\,(\Phi(x)\Phi(y) + \Phi(y)\Phi(x)), \quad (14)$$

both of which are proportional to $\Delta(x-y)$ and thus have support within the light-cone. Indeed the commutator of any two (Wick ordered) monomials of the fundamental fields and a finite number of derivatives can be written as a sum of terms proportional to $\Delta(x-y)$ and derivatives thereof, and so these will also have light-cone support. So operators formed by monomials and a finite number of derivatives commute at space-like distances. (It is well known that the converse is also true [11,12].)

Problems occur however if one considers monomials containing an infinite number of derivatives such as $\Phi * \Phi(x)$. For example

$$[\Phi(x), \Phi * \Phi(y)] = \mathrm{i}\Delta(x-y) *_y \Phi(y) + \mathrm{i}\Phi(y) *_y \Delta(x-y), \quad (15)$$

and the presence of the star can spoil the support properties of the commutator. This commutator consists of four terms similar to

$$\begin{aligned}&\mathrm{i}\Delta^+(x-y) *_y \Phi^+(y) + \mathrm{i}\Phi^+(y) *_y \Delta^+(x-y)\\ &= \frac{1}{(2\pi)^{9/2}}\int \frac{\mathrm{d}^3k\mathrm{d}^3k'}{4k^0k'^0}\mathrm{e}^{-\mathrm{i}k^+y}A(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k'^+(x-y)}\\ &\quad\times\left(\mathrm{e}^{-\mathrm{i}k'^+\wedge k^+} + \mathrm{e}^{-\mathrm{i}k^+\wedge k'^+}\right)\\ &= \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{2k^0}\mathrm{e}^{-\mathrm{i}k^+y}A(\boldsymbol{k}) \quad (16)\\ &\quad\times\left(\Delta^+(x-y+\tilde{k}/2) + \Delta^+(x-y-\tilde{k}/2)\right),\end{aligned}$$

where

$$\begin{aligned}&*_y = \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu}, \quad k\wedge k' = \frac{1}{2}k^\mu\theta_{\mu\nu}k'^\nu,\\ &\tilde{k}^\mu = \theta^{\mu\nu}k^+_\nu, \quad k^+ = (\omega_k, \boldsymbol{k}). \quad (17)\end{aligned}$$

The commutator is no longer proportional to $\Delta(x-y)$ but is shifted by an amount depending on k which is integrated over. Thus in general the commutator no longer has support only within the light-cone, and the operators do not commute at space-like distances. In general equal time commutators will not vanish.

Note however that in the special case $\theta_e = 0$ then $\tilde{k}^0 = \tilde{k}^1 = 0$ and so the shift only occurs in the x^2, x^3 direction. In this case equal time commutators still vanish and the support properties become "wedge-like" (see for example [13]). Note that it is also possible to define the free theory so that this also has only wedge-like support properties and only has the symmetry $SO(1,1)\times SO(2)$ but not the full four-dimensional Lorentz group [14].

3.2 Interaction – tree approximation: symmetry breaking

We define time ordered Green functions via the Gell-Mann–Low formula:

$$\langle T\Phi(x_1)\dots\Phi(x_n)\rangle = \left\langle T\Phi(x_1)\dots\Phi(x_n)\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0, \quad (18)$$

where S_{int} is the interaction part of the action. On the left-hand side of this equation we have interacting fields and the expectation value is with respect to the interacting vacuum whereas on the right-hand side we take free fields and the free vacuum which we indicate by the subscript 0.

Usually the time ordered product of two operators O_1, O_2 is defined as

$$\begin{aligned}&TO_1(x)O_2(y) \quad (19)\\ &= \theta(x^0-y^0)O_1(x)O_2(y) + \theta(x^0-y^0)O_2(y)O_1(x).\end{aligned}$$

Now $\theta(x^0-y^0)$ is not Lorentz invariant. However the time ordered product defined by (19) *is* Lorentz invariant provided $O_1(x)$ and $O_2(y)$ commute at space-like distances. This is because if $x-y$ is time-like then $\theta(x^0-y^0)$ is Lorentz invariant whereas if $x-y$ is space-like then the order is irrelevant and

$$\begin{aligned}TO(x)O(y) &= (\theta(x^0-y^0) + \theta(x^0-y^0))O_1(x)O_2(y)\\ &= O_1(x)O_2(y), \quad (20)\end{aligned}$$

which is also Lorentz invariant.

This is no longer true in a general non-commutative field theory since, as we saw in the previous section, equal time commutators no longer vanish (unless we consider the special case with $\theta_e = 0$). So if we use the above definition of time ordering we do not expect the time ordered products to obey even the remaining $SO(1,1)\times SO(2)$ invariance. We therefore introduce a new definition of time ordering.

3.3 Light-wedge variables and new time ordering

In the present case of $SO(1,1)\times SO(2)$ invariance a suitably adapted time ordering will be defined in the next subsection; hence we introduce the respective variables as

$$u = (x^0 - x^1)/\sqrt{2},\ v = (x^0 + x^1)/\sqrt{2}. \quad (21)$$

It is useful to re-express the free fields in terms of these variables which we call "light-wedge variables". Note that such co-ordinates are used in light-cone quantization of field theories. We define momenta (with indices downstairs) as

$$k_u = \frac{k_0 - k_1}{\sqrt{2}},\ k_v = \frac{k_0 + k_1}{\sqrt{2}}, \quad (22)$$

and this means $k^u = k_v,\ k^v = k_u$. So the solution of the Klein–Gordon equation (this is completely equivalent to the usual solution via a change of variables) becomes

$$\Phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^4k\ \delta(2k_uk_v - k_ak_a - m^2)A(k)\mathrm{e}^{-\mathrm{i}kx} \quad (23)$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{\mathrm{d}^3k}{2k_v}A(\mathbf{k})\mathrm{e}^{-\mathrm{i}\overline{k}x}, \quad (24)$$

where in the second line $\mathrm{d}^3k := \mathrm{d}k_v\mathrm{d}k_2\mathrm{d}k_3$ and $\mathbf{k} = (k_v, k_a)$ and the on-shell momentum $\overline{k}$ is defined as

$$\overline{k}_u = (m^2 + k_ak_a)/(2k_v), \quad \overline{k}_v = k_v, \quad \overline{k}_a = k_a, \quad a = 2,3. \quad (25)$$

Note that with these variables there is no need to separate positive and negative frequency parts. Taking k_v positive corresponds to positive frequency and vice versa. The reality of Φ implies

$$A^\dagger(\mathbf{k}) = -A(-\mathbf{k}). \tag{26}$$

Inverting (24) we can express $A(\mathbf{k})$ in terms of the field $\Phi(x)$:

$$A(\mathbf{k}) = \frac{1}{2\pi^{3/2}} \int \mathrm{d}^3x 2k_v \mathrm{e}^{\mathrm{i}\bar{k}x} \Phi(x). \tag{27}$$

We quantize the fields with the commutation relation

$$[A(\mathbf{k}), A(\mathbf{k}')] = 2k_v \delta^3(\mathbf{k} + \mathbf{k}'), \tag{28}$$

and the vacuum satisfies

$$A(\mathbf{k})|0\rangle = 0, \quad k_v < 0, \qquad \langle 0|A(\mathbf{k}) = 0, \quad k_v > 0. \tag{29}$$

The commutator function has the form

$$\mathrm{i}\Delta(x - x') = [\Phi(x), \Phi(x')] = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3k}{2k_v} \mathrm{e}^{-\mathrm{i}\bar{k}(x-x')}, \tag{30}$$

and Δ^+ (Δ^-) are given by similar expressions but with the k_v integration restricted to the interval $(0,\infty)$ ($(-\infty, 0)$). One can explicitly check using Bessel function identities such as (in the two-dimensional case)

$$(\pi\mathrm{i}/2) H_0^{(1)}(-2m\sqrt{uv}) = \int_0^\infty \frac{\mathrm{d}k_v}{2k_v} \mathrm{e}^{-im/k_u(u)+k_v(v)} \tag{31}$$

that these give the same expressions as in the usual case.

Finally, we will wish to redefine the causal Green function according to the new time ordering as

$$\Delta_\mathrm{c}(x) = \theta(u)\Delta^+(x) - \theta(-u)\Delta^-(x). \tag{32}$$

In fact this is identical to the standard propagator defined in terms of the usual time ordering as we argue below.

3.4 $SO(1,1) \times SO(2)$ invariant time ordering

In order to keep the remaining $SO(1,1) \times SO(2)$ symmetry we use the following definition of time ordering:

$$TO_1(x_1)O_2(x_2) = \theta(u_1 - u_2)O_1(x_1)O_2(x_2) + \theta(u_2 - u_1)O_2(x_2)O_1(x_1), \tag{33}$$

where we have used the "light-wedge" coordinates u, v defined in (21).

Note that for two space-like commuting operators this time ordering is in fact equivalent to the usual one since for time-like x, we have that $u > 0 \Leftrightarrow x^0 > 0$. So we are using a choice of time ordering which is equivalent to the usual prescription for ordinary theories, but which also maintains the $SO(1,1) \times SO(2)$ symmetry in the non-commutative case. This is one way of seeing that the free propagator defined with the u time ordering is equivalent to the usual time ordering, since the propagator used in ordinary perturbation theory is just the vacuum expectation value of the time ordered product of two fundamental fields which are free and do indeed commute at space-like distances.

To see that this new time ordering respects the symmetry note that under a $SO^+(1,1)$ transformation $u \to au,\ v \to v/a; a > 0$ and so $\theta(u)$ is invariant without the need for space-like commutativity.

4 The quantum equation of motion

4.1 Usual time ordering

We wish to consider the tree-level quantum equations of motion for a non-commutative field theory. Eventually we will consider an interaction term ϕ_*^3 but to illustrate the techniques we first consider some simple cases. In this subsection we define time ordering in the standard way with respect to x^0 whereas in the next subsection we will use the new time ordering with respect to u. Firstly consider the standard case of a theory defined by a free Lagrangian together with an interaction Lagrangian which contains no time derivatives. We wish to find the quantum equation of motion for such a theory, i.e. the equation of motion for a field inserted into a Green's function defined using the Gell-Mann–Low formula

$$\begin{aligned}
&(\Box_x + m^2) \langle T\phi(x)X\rangle \\
&= (\Box_x + m^2) \left\langle T\phi(x)X\mathrm{e}^{\mathrm{i}S_\mathrm{int}}\right\rangle_0 && (34)\\
&= (\Box_x + m^2) \int \mathrm{d}^4y \,\langle T\phi(x)\phi(y)\rangle_0 \\
&\qquad \times \left\langle T\frac{\delta}{\delta\phi(y)}(X\mathrm{e}^{\mathrm{i}S_\mathrm{int}})\right\rangle_0 && (35)\\
&= \left\langle T\frac{\delta S_\mathrm{int}}{\delta\phi(x)}X\mathrm{e}^{\mathrm{i}S_\mathrm{int}}\right\rangle_0 + \text{c.t.} && (36)\\
&= \left\langle T\frac{\delta S_\mathrm{int}}{\delta\phi(x)}X\right\rangle + \text{c.t.} && (37)
\end{aligned}$$

Here X represents any monomial of fields and derivatives thereof. The third line is obtained using Wick's theorem and we will discuss this further below: it is only valid as written when there are no time derivatives in S_int. In the fourth line we have used that

$$(\Box_x + m^2) \langle T(\phi(x)\phi(y))\rangle_0 = -\mathrm{i}\delta(x - y), \tag{38}$$

and "c.t." stands for "contact terms" which arise from $\frac{\delta}{\delta\phi(y)}X$. Finally we re-express the answer in terms of interacting fields to obtain the fifth line. We find a quantum equation similar to the classical equation up to contact terms.

As mentioned (35) can be derived from Wick's theorem but only when S_int contains no time derivatives: these interfere with the time ordering. To see this consider for illustration S_int of the form

$$S_\mathrm{int} = g \int \mathrm{d}^4x \,\mathcal{O}\,(\partial_0)^n\phi(x), \tag{39}$$

where $\mathcal{O}$ is a monomial in ϕ. Then using

$$(\Box_x + m^2)\,\langle T\,(\phi(x)(\partial_0)^n\phi(y))\rangle_0 = \begin{cases} -\mathrm{i}(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n}{2}}\,\delta(x-y), & n \text{ even}, \\ \mathrm{i}\partial_0(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n-1}{2}}\,\delta(x-y), & n \text{ odd}, \end{cases} \tag{40}$$

we find that the quantum equation of motion for n even is

$$(\Box_x + m^2)\,\langle T(\phi(x)X)\rangle \tag{41}$$
$$= (\Box_x + m^2)\,\mathrm{i}\int \mathrm{d}^4y\,\langle T\phi(x)(\partial_0)^n\phi(y)\rangle_0\,\left\langle T\mathcal{O}X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0$$
$$+ (\Box_x + m^2)\,\mathrm{i}\int \mathrm{d}^4y\,\langle T\phi(x)\phi(y)\rangle_0 \tag{42}$$
$$\times\left\langle T\frac{\partial\mathcal{O}}{\partial\phi}(y)(\partial_0)^n\phi(y)X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0$$
$$= g(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n}{2}}\,\left\langle T\left(\mathcal{O}X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right)\right\rangle_0$$
$$+g\left\langle T\left(\frac{\partial\mathcal{O}}{\partial\phi}(\partial_0)^n\phi(x)X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right)\right\rangle_0 + \text{c.t.} \tag{43}$$
$$= g\left\langle T\left((\boldsymbol{\partial_x}^2 - m^2)^{\frac{n}{2}}\mathcal{O}(x)X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right)\right\rangle_0 \tag{44}$$
$$+g\left\langle T\left(\frac{\partial\mathcal{O}}{\partial\phi}(x)(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n}{2}}\phi(x)X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right)\right\rangle_0 + \text{c.t.}$$
$$= \left\langle T\left(\frac{\delta\tilde{S}_{\mathrm{int}}}{\delta\phi(x)}X\right)\right\rangle + \text{c.t.}, \tag{45}$$

where we define a modified effective interaction as

$$\tilde{S}_{\mathrm{int}} = g\int \mathrm{d}^4x\mathcal{O}(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n}{2}}\phi(x), \quad n \text{ even}. \tag{46}$$

Notice that if ϕ is a free field $S(\phi) = \tilde{S}(\phi)$ but for a general field the two actions are different. Thus the manipulations from (41) to (44) work because there we are dealing with free fields, as indicated by the subscript 0 for the correlators (see (18)). But the result for interacting fields of (45) is non-trivial.

Some comments on the manipulations above. Equation (41) is obtained via the Gell-Mann–Low formula using Wick's theorem (and is the analogue of (35)). To obtain (43) we have used the first equation of (40) and integrated out the delta function. On going from (43) to (44), in the first term we have moved the differential operator inside the correlation function (which is allowed since there are no time derivatives) and in the second term we have used the equations of motion for the free field sitting in the propagator.

A crucial point is that we take all derivatives occurring in the interaction Lagrangian to act *before* the time ordering whereas the integration itself is taken after the time ordering. It is also possible to define a time ordering T_* where all derivatives occur outside the time ordering and this definition gives the naïve Feynman rules. For a standard quantum field theory these two definitions differ by local terms only and are therefore equivalent after a finite renormalisation whereas for a non-commutative field theory the equivalence is not to be expected.

The case with n odd is more complicated. In this case the quantum equation of motion is

$$(\Box_x + m^2)\,\langle T\phi(x)X\rangle \tag{47}$$
$$= -g\,\partial_0(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n-1}{2}}\,\left\langle T\mathcal{O}X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0$$
$$+g\left\langle T\frac{\partial\mathcal{O}}{\partial\phi}(\partial_0)^n\phi(x)X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0 + \text{c.t.} \tag{48}$$
$$= -g\,\partial_0\left\langle T(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n-1}{2}}\mathcal{O}X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0 \tag{49}$$
$$+g\left\langle T\frac{\partial\mathcal{O}}{\partial\phi}\partial_0(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n-1}{2}}\phi(x)X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0 + \text{c.t.}$$

Here there is a crucial difference to the case where we have an even number of time derivatives in the interaction Lagrangian. We wish to rewrite this as $\left\langle T\left(\frac{\delta\tilde{S}_{\mathrm{int}}}{\delta\phi(x)}X\right)\right\rangle +$ c.t. with the modified action

$$\tilde{S}_{\mathrm{int}} = \int \mathrm{d}^4x\,\mathcal{O}\,\partial_0(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n-1}{2}}\phi(x), \quad n \text{ odd}. \tag{50}$$

In (49), however, one term has the time derivative acting after the time ordering and one has it acting before the time ordering. If we wish to write this in terms of a modified action (with all derivatives acting after the time ordering) then we pick up an additional second order term involving a commutator at second order in the coupling. This extra term comes from pulling the time derivative outside the time ordering and has the form

$$-\mathrm{i}g^2\int \mathrm{d}^4y\,\delta(x^0 - y^0) \tag{51}$$
$$\times\left\langle T\left[\frac{\partial\mathcal{O}}{\partial\phi}(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n-1}{2}}\phi(x), \mathcal{O}(y)(\partial_0)^n\phi(y)\right]X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0$$
$$= +g^2\left\langle T\left\{(\boldsymbol{\partial_x}^2 - m^2)^{n-1}\left(\mathcal{O}\frac{\partial\mathcal{O}}{\partial\phi}\right)\right.\right.$$
$$\left.\left.+(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n-1}{2}}\left(\mathcal{O}\frac{\partial^2\mathcal{O}}{\partial\phi^2}(\boldsymbol{\partial_x}^2 - m^2)^{\frac{n-1}{2}}\phi\right)\right\}X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0$$

arising from the time derivative acting on the time ordering.

In a standard quantum field theory, with only a finite number of time derivatives, one can remove this extra term using the method of finite counter terms, and this is equivalent to using the $T*$.

Note that the above result shows that, using the Gell-Mann–Low formula, two Lagrangians which differ by total derivatives (and hence give the same action) can nevertheless lead to different quantum equations of motion. For example, consider the case $n = 1$, $\mathcal{O} = \phi^m$; then the interaction Lagrangian is a total derivative $L = \phi^m\partial_0\phi = \partial_0\phi^{m+1}/(m+1)$ and so the action is the same as the free one. However the quantum equation of motion obtained using the Gell-Mann–Low formula is not the same as the free one. In this case (49) reads

$$(\Box_x + m^2)\,\langle T\phi(x)X\rangle \tag{52}$$
$$= -g\,\partial_0\left\langle T\phi(x)^m X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0 + g\left\langle T\partial_0\phi(x)^m X\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle_0 + \text{c.t.}$$

and the two terms on the right-hand side do not cancel because one time derivative is inside the time ordering and one outside. We obtain the non-vanishing term (51). However once again in a local theory with a finite number of time derivatives, these discrepancies can be removed using finite counter terms. In a theory defined with the star product however this discrepancy may be unavoidable. Note that the fact that with the standard time ordering Lagrangians which differ by total derivatives can lead to different quantum theories has also been noted in [3]. It turns out that this is not the case for the new time ordering which we use in the next section and can thus be seen as a further advantge of this over the standard time ordering.

This method can be extended to more general actions. The prescription is simple: for the quantum equation of motion, we obtain a modified action $\tilde{S}_{\rm int}$ simply by replacing every occurrence of ∂_t^2 with $\boldsymbol{\partial_x}^2 - m^2$ and leaving behind a single ∂_t if necessary.

So in particular for the ϕ_*^3 theory we *almost* obtain the quantum action simply by replacing the Moyal star with the following (in momentum space):

$$\mathrm{e}^{-\mathrm{i}p\wedge q} \to \cos(p^+ \wedge q^+) - \mathrm{i}\sin(p^+ \wedge q^+)\frac{p\wedge q}{p^+ \wedge q^+}, \qquad (53)$$

where $p^+ = (\sqrt{\boldsymbol{p}^2 + m^2}, \boldsymbol{p})$. This is simply what one obtains in momentum space by replacing every occurrence of ∂_0^2 with $\boldsymbol{\partial_x}^2 - m^2$, and leaving behind a single ∂_0 if you started with an odd number. But as in the example above, one has to be careful about whether the time derivatives act before or after the time ordering. Those time derivatives which act before the time ordering must be pulled out of the time ordering, leading to an additional term at order g^2 (as we saw in (51)). Furthermore in the case of non-commutative field theory this additional term is not $SO(1,1) \times SO(2)$ invariant since it involves an integral of $\delta(x^0 - y^0)[\phi(x), \phi_*^3(y)]$. The commutator in this case does not give a space-like delta function needed in order to complete the expression into a Lorentz invariant delta function as occurs for interaction terms involving only finite numbers of time derivatives[1].

4.2 New time ordering

We now repeat the above calculation using the time ordering adapted to the $SO(1,1)$ symmetries. We expect this case to preserve the remaining symmetries for the reasons given previously. Green's functions are defined by the Gell-Mann–Low formula (18) together with the time ordering defined with the u coordinate as in (33). We wish to calculate the quantum equation of motion as we did in the previous section for the usual time ordering. In the case of an interaction Lagrangian containing no explicit u-derivatives there will be no interference with the time ordering and the quantum equation of motion will reproduce that of the classical one; that is, (37) will be satisfied. When the interaction Lagrangian contains u-derivatives however this will interfere with the time ordering just as time derivatives did in the previous section.

Consider the interaction Lagrangian

$$S_{\rm int} = g \int \mathrm{d}^4 x\, \mathcal{O}\, (\partial_u)^n \phi(x); \qquad (54)$$

then using

$$\begin{aligned} &(\Box_x + m^2) \left\langle T\left(\phi(x)(\partial_u')^n \phi(x')\right)\right\rangle_0 \\ &= -\mathrm{i}\delta(u-u') \int \mathrm{d}^3 k (\mathrm{i}\bar{k}_u)^n \mathrm{e}^{-\mathrm{i}\bar{k}(x-x')} \qquad (55)\\ &= -\mathrm{i}\left(\frac{m^2 - \partial_2^2 - \partial_3^2}{2\partial_v}\right)^n \delta(x-x'), \qquad (56) \end{aligned}$$

where $\mathrm{d}^3 k = \mathrm{d}k_v \mathrm{d}k_2 \mathrm{d}k_3$, we find the quantum equation of motion:

$$\begin{aligned} &(\Box_x + m^2)\langle T(\phi(x)X)\rangle \qquad (57)\\ &= g(-1)^n \left(\frac{\partial_2^2 + \partial_3^2 - m^2}{2\partial_v}\right)^n \left\langle T\mathcal{O}X\mathrm{e}^{\mathrm{i}S_{\rm int}}\right\rangle_0 \\ &\quad + g\left\langle T\frac{\partial \mathcal{O}}{\partial\phi}(\partial_u)^n \phi(x) X \mathrm{e}^{\mathrm{i}S_{\rm int}}\right\rangle_0 + \text{c.t.} \qquad (58)\\ &= g\left\langle T(-1)^n \left(\frac{\partial_2^2 + \partial_3^2 - m^2}{2\partial_v}\right)^n \mathcal{O}X\mathrm{e}^{\mathrm{i}S_{\rm int}}\right\rangle_0 \qquad (59)\\ &\quad + g\left\langle T\frac{\partial \mathcal{O}}{\partial\phi}\left(\frac{\partial_2^2 + \partial_3^2 - m^2}{2\partial_v}\right)^n \phi(y) X\mathrm{e}^{\mathrm{i}S_{\rm int}}\right\rangle_0 + \text{c.t.} \\ &= \left\langle T\frac{\delta\tilde{S}_{\rm int}}{\delta\phi(x)} X\right\rangle + \text{c.t.}, \qquad (60) \end{aligned}$$

where we define a modified effective action as

$$\tilde{S}_{\rm int} = g\int \mathrm{d}^4 x \mathcal{O}\left(\frac{\partial_2^2 + \partial_3^2 - m^2}{2\partial_v}\right)^n \phi(x). \qquad (61)$$

Note that there is here no distinction between n odd and n even, and remarkably there is no complication with left over time- (i.e. u-) derivatives acting both inside and outside the time ordering which were the origin of the breaking of Lorentz invariance in the previous case (recall that with the usual time ordering, for n odd we were left with a remaining ∂_0 which gave extra terms and led to $SO(1,1)$ violating terms in the non-commutative case.) Here all u-derivatives have disappeared and so the derivative operators can be happily commuted through the time ordering.

Notice that to go from (58) to (59) in the first term we have used the fact that we can put the differential operator inside the correlator since it commutes with the time ordering and in the second term we have used the fact that we are dealing with free fields in replacing ∂_u^n. In the case of the new time ordering the modified effective action is obtained by simply replacing every occurrence of ∂_u by $(\partial_2^2 + \partial_3^2 - m^2)/2\partial_v$ in the action. This is because the Gell-Mann–Low formula gives time ordered vacuum

[1] The breaking of the remaining symmetries in TOPT for non-commutative field theory was first pointed out by Reichenbach.

expectation values of interacting fields in terms of those for free fields for which this replacement is possible by the free field equations of motion.

Indeed a quicker way to deal with the complications in arriving at a quantum equation of motion due to the time ordering is to argue as follows. Firstly use the Gell-Mann–Low formula to obtain an equation similar to (34). Now remove all u-derivatives using the free field equation of motion (since on the RHS of (34) we have free fields.) This essentially involves replacing S with $\tilde{S}$. Now simply follow the arguments leading to (37) which are now valid since the interaction Lagrangian has no time derivatives. Clearly we end up with a quantum equation of motion involving the modified Lagrangian $\tilde{S}$.

It may appear at first sight from this argument that the modified action is the same as the original action and we are free to use either. It should be noted however that the resulting modified effective action is for vacuum expectation values of *interacting fields* (i.e. (57) and (60) are for interacting fields) and so $\tilde{S} \neq S$. Indeed we will later be able to define Feynman rules using the modified interaction and this is only possible once all time derivatives have been removed.

Note that in the above formulae we define the inverse differential $1/\partial_x$ via its Fourier transform as

$$\frac{1}{\partial_x} f(x) = \int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kx} \frac{\tilde{f}(k)}{-\mathrm{i}k} \tag{62}$$

and integration by parts follows straightforwardly:

$$\int \mathrm{d}x \frac{1}{\partial_x} f(x) g(x) = -\int \mathrm{d}x f(x) \tfrac{1}{\partial_x} g(x). \tag{63}$$

In non-commutative ϕ^3_* field theory therefore we simply change the Moyal star in momentum space as follows:

$$* = \mathrm{e}^{-\frac{\mathrm{i}}{2} p \wedge q} \to \overline{*} = \mathrm{e}^{-\frac{\mathrm{i}}{2} \overline{p} \wedge \overline{q}} \tag{64}$$

where $\overline{p}$ is the on-shell momentum defined in (25) to obtain a modified action $\tilde{S}$, whose naïve variation leads to the quantum equations of motion.

4.3 The meaning of $\tilde{S}$

We have shown that

$$\left\langle T \frac{\delta \tilde{S}}{\delta \phi} X \right\rangle = \mathrm{c.t.} \tag{65}$$

where $\tilde{S} = \int \mathrm{d}^4 x \left(\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \right) + \tilde{S}_{\mathrm{int}}$ and we interpret all u-derivatives to be acting outside the time ordering. This can be rewritten in terms of the generating functional for the Green's functions Z as

$$-(\Box + m^2) \frac{\delta}{\delta J(x)} Z + \left. \frac{\delta \tilde{S}_{\mathrm{int}}}{\delta \phi} \right|_{\phi = \frac{\delta}{\mathrm{i}\delta J}} Z = \mathrm{i} J Z, \tag{66}$$

where the right-hand side gives the contribution of the contact terms.

The generator of connected Green's functions Z_{c} is defined by $Z = \mathrm{e}^{\mathrm{i} Z_{\mathrm{c}}}$ and we define the one-point function $\phi_{\mathrm{c}}(x) = \mathrm{i} \frac{\delta Z_{\mathrm{c}}}{\delta J(x)}$. Equation (66) then becomes (at tree level)

$$(\Box + m^2) \phi_{\mathrm{c}} + \left. \frac{\delta \tilde{S}_{\mathrm{int}}}{\delta \phi} \right|_{\phi = \phi_{\mathrm{c}}} = J. \tag{67}$$

The above equation ignores all terms of the form $\frac{\delta^n}{\delta J^n} Z_{\mathrm{c}}$ for $n > 1$. Such terms involve at least $n-1$ closed loops and therefore vanish in the tree approximation. The generator of one-particle irreducible diagrams Γ is then defined in the usual way as a functional of ϕ_{c}: $\Gamma = Z_{\mathrm{c}} - \int \mathrm{d}x J \phi_{\mathrm{c}}$ so that Γ satisfies

$$\frac{\delta \Gamma}{\delta \phi_{\mathrm{c}}} = -J. \tag{68}$$

In particular, at tree level we have

$$\frac{\delta \Gamma}{\delta \phi_{\mathrm{c}}} = -J = -(\Box + m^2) \phi_{\mathrm{c}} + \left. \frac{\delta \tilde{S}_{\mathrm{int}}}{\delta \phi} \right|_{\phi = \phi_{\mathrm{c}}} = \left. \frac{\delta \tilde{S}}{\delta \phi} \right|_{\phi = \phi_{\mathrm{c}}}; \tag{69}$$

that is,

$$\Gamma(\phi) = \tilde{S}(\phi), \tag{70}$$

or in other words, $\tilde{S}$ is the tree-level effective action.

This is a somewhat remarkable result: usually the zero-loop approximation to Γ can be identified with the classical action. And the vertices of the classical action are used as the vertices in the interaction as defined, say via the Gell-Mann–Low formula. Here however the zero-loop approximation to the vertex functional Γ cannot be identified with the classical action but differs by the transition to the mass-shell within the star product vertices as enforced by the quantization procedure. It is also to be noted that this on-shell star product is not really a star product: it is e.g. not associative.

5 Symmetries

We wish to prove explicitly that the theory defined via the Gell-Mann–Low formula and with the modified time ordering is invariant under translations and the remaining $SO(1,1) \times SO(2)$ symmetry (at least at tree level).

For this we simply have to show that the effective action Γ is invariant under these symmetries. If it is, then we will also be able to construct conserved energy- and angular-momentum tensors by Noether's theorem.

5.1 Translations: the energy momentum tensor

Since the effective action Γ has no explicit x dependence, it must be translation invariant. We do not consider here the free part of the effective action which takes the standard form and gives the standard energy momentum tensor. The interaction part has the form

$$\begin{aligned}\Gamma_{\text{int}} &= \int \mathrm{d}x \phi_1 \bar{*} \phi_2 \bar{*} \phi_3 \qquad (71)\\ &= \int \mathrm{d}\mu(p_i)\ \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3) F(\overline{p}_1, \overline{p}_2, \overline{p}_3),\end{aligned}$$

with $\mathrm{d}\mu(p_i) := \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3 \delta(p_1+p_2+p_3)$ and where $\tilde{\phi}$ is the Fourier transform of ϕ, and F is the non-commutative phase factor

$$F(p_1, p_2, p_3) = \mathrm{e}^{-\mathrm{i}(p_1 \wedge p_2 + p_1 \wedge p_3 + p_2 \wedge p_3)}. \qquad (72)$$

Recall that $\bar{*}$ is defined (in momentum space) in (64) and $\overline{p}$ is the on-shell momentum defined in (25). Explicitly, the Ward identity for infinitesimal translations has the form

$$\begin{aligned}\delta\Gamma &= \int \mathrm{d}x (a^\mu \partial_\mu \phi) \bar{*} \phi \bar{*} \phi \qquad (73)\\ &\qquad + \phi \bar{*} (a^\mu \partial_\mu \phi) \bar{*} \phi + \phi \bar{*} \phi \bar{*} (a^\mu \partial_\mu \phi)\\ &= \int \mathrm{d}\mu(q, p_i)\ \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{a}^\mu(q)(-\mathrm{i})\\ &\qquad \times I_\mu(q, p_1, p_2, p_3), \qquad (74)\end{aligned}$$

with $\mathrm{d}\mu(q, p_i) := \mathrm{d}q \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3 \delta(q+p_1+p_2+p_3)$ and where

$$\begin{aligned}&I_\mu(q, p_1, p_2, p_3)\\ &= p_{1\mu} F(\overline{q+p_1}, \overline{p}_2, \overline{p}_3) \qquad (75)\\ &\qquad + p_{2\mu} F(\overline{p}_1, \overline{q+p_2}, \overline{p}_3) + p_{3\mu} F(\overline{p}_1, \overline{p}_2, \overline{p_3+q})\\ &= (p_1+p_2+p_3)_\mu\ F(\overline{p}_1, \overline{p}_2, \overline{p}_3)\\ &\qquad + p_{1\mu}(F(\overline{q+p_1}, \overline{p}_2, \overline{p}_3) - F(\overline{p}_1, \overline{p}_2, \overline{p}_3))\\ &\qquad + p_{2\mu}(F(\overline{p}_1, \overline{q+p_2}, \overline{p}_3) - F(\overline{p}_1, \overline{p}_2, \overline{p}_3))\\ &\qquad + p_{3\mu}(F(\overline{p}_1, \overline{p}_2, \overline{q+p_3}) - F(\overline{p}_1, \overline{p}_2, \overline{p}_3)). \qquad (76)\end{aligned}$$

Now

$$\begin{aligned}&F(\overline{p_1+q}, \overline{p}_2, \overline{p}_3) - F(\overline{p}_1, \overline{p}_2, \overline{p}_3)\\ &\sim F(\overline{p}_1, \overline{p}_2, \overline{p}_3) \times \Phi_1 \times (-\mathrm{i}\, q_1 \wedge (\overline{p}_2 + \overline{p}_3)), \qquad (77)\\ &F(\overline{p}_1, \overline{p_2+q}, \overline{p}_3) - F(\overline{p}_1, \overline{p}_2, \overline{p}_3)\\ &\sim F(\overline{p}_1, \overline{p}_2, \overline{p}_3) \times \Phi_2 \times (-\mathrm{i}\, q_2 \wedge (\overline{p}_3 - \overline{p}_1)), \qquad (78)\\ &F(\overline{p}_1, \overline{p}_2, \overline{p_3+q}) - F(\overline{p}_1, \overline{p}_2, \overline{p}_3)\\ &\sim F(\overline{p}_1, \overline{p}_2, \overline{p}_3) \times \Phi_3 \times (\mathrm{i}\, q_3 \wedge (\overline{p}_1 + \overline{p}_2)), \qquad (79)\end{aligned}$$

where

$$\Phi_1 := \left(\frac{\mathrm{e}^{\mathrm{i}(\overline{p_2+p_3}+\overline{p}_1)\wedge(\overline{p}_2+\overline{p}_3)} - 1}{\mathrm{i}(\overline{p_2+p_3}+\overline{p}_1)\wedge(\overline{p}_2+\overline{p}_3)} \right), \qquad (80)$$

$$\Phi_2 := \left(\frac{\mathrm{e}^{\mathrm{i}(\overline{p_1+p_3}+\overline{p}_2)\wedge(\overline{p}_3-\overline{p}_1)} - 1}{\mathrm{i}(\overline{p_1+p_3}+\overline{p}_2)\wedge(\overline{p}_3-\overline{p}_1)} \right), \qquad (81)$$

$$\Phi_3 := \left(\frac{\mathrm{e}^{-\mathrm{i}(\overline{p}_1+\overline{p_2+p_3})\wedge(\overline{p}_1+\overline{p}_2)} - 1}{-\mathrm{i}(\overline{p}_1+\overline{p_2+p_3})\wedge(\overline{p}_1+\overline{p}_2)} \right). \qquad (82)$$

We have here defined

$$q_i := \overline{q+p_i} - \overline{p}_i, \qquad (83)$$

so three of the components of q_i are equal to those of q i.e. $(q_i)_v = q_v$, $(q_i)_2 = q_2$, $(q_i)_3 = q_3$ whereas the uth component is for example

$$\begin{aligned}(q_1)_u &= \left(\frac{m^2 + (q+p_1)_a (q+p_1)_a}{2(q_v + p_{1v})} - \frac{m^2 + p_a p_a}{2p_{1v}} \right)\\ &\sim \frac{q_v(m^2 + p_{1a}p_{1a}) - q_a(p_1 - p_2 - p_3)_a p_{1v}}{2(p_{2v} + p_{3v})p_{1v}}. \qquad (84)\end{aligned}$$

The "$\sim$" in all the above equations means "equal when multiplied by $\delta(q+p_1+p_2+p_3)$": we have used the delta function to ensure that we have an expression which is linear in q (corresponding to a single derivative of a_μ.) Now define $S_{i\mu}$ via

$$\begin{aligned}q_1 \wedge (\overline{p}_2 + \overline{p}_3) &= q^\mu S_{1\mu}, \qquad (85)\\ q_2 \wedge (\overline{p}_3 - \overline{p}_1) &= q^\mu S_{2\mu}, \qquad (86)\\ -q_3 \wedge (\overline{p}_1 + \overline{p}_2) &= q^\mu S_{3\mu}. \qquad (87)\end{aligned}$$

In this way we have obtained an expression for I_μ which is linear in q

$$I_\mu \sim F(\overline{p}_1, \overline{p}_2, \overline{p}_3) \left(-q_\mu - \mathrm{i}\, q_\nu \sum_i S_i^\nu p_{i\mu} \Phi_i \right). \qquad (88)$$

Putting this into (74) gives

$$\delta\Gamma = -\int \mathrm{d}x \partial_\nu a^\mu(x) T_\mu^\nu, \qquad (89)$$

where

$$\begin{aligned}(T_{\text{int}})_\mu^\nu &= \delta_\mu^\nu \Gamma_{\text{int}}\\ &+ \mathrm{i} \int \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3 \mathrm{e}^{-\mathrm{i}x(p_1+p_2+p_3)} \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\\ &\qquad \times F(\overline{p}_1, \overline{p}_2, \overline{p}_3) \sum_i S_i^\nu p_{i\mu} \Phi_i \qquad (90)\end{aligned}$$

for $a, b \in \{2, 3\}$.

5.2 Lorentz transformations: angular-momentum tensor

The effective action is also invariant under the remaining $SO(1,1) \times SO(2)$ transformation. The underlying reason why the effective action is invariant under these symmetries is that the symmetries commute with the projection of p onto the mass-shell $p \to \overline{p}$. In other words we have $\overline{p + \delta p} = \overline{p} + \delta\overline{p}$ where δ is an infinitesimal $SO(1,1) \times SO(2)$ transformation. We show this explicitly in (95)

Note that there is another crucial difference with the standard time ordering here. With the standard time ordering one projects onto the mass-shell by replacing p_0 with $\pm\sqrt{p_1^2 + p_a p^a + m^2}$ instead of replacing p_u as we do with the new time ordering. This projection does not commute with the $SO(1,1) \times SO(2)$ transformation, thus leading to a loss of the symmetry.

The explicit proof of covariance of the effective action and construction of the angular-momentum tensor follows in a way similar to that of the energy-momentum tensor. An infinitesimal Lorentz transformation has the form (73) with $a^\mu = w^\mu_\nu x^\nu$ and so $\delta\Gamma$ can be written

$$\delta\Gamma = \int \mathrm{d}\mu(p_i, q)\ \tilde{w}^{\mu\nu}(q) \times \left(p_{1\nu}\partial_\mu\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)F(\overline{q+p_1}, \overline{p}_2, \overline{p}_3) + \dots\right), \tag{91}$$

where the dots indicate two more similar terms. We proceed as for the energy-momentum tensor we write $F(\overline{q+p_1}, \overline{p}_2, \overline{p}_3)$ as $F(\overline{p}_1, \overline{p}_2, \overline{p}_3) + (F(\overline{q+p_1}, \overline{p}_2, \overline{p}_3) - F(\overline{p}_1, \overline{p}_2, \overline{p}_3))$ and similarly for the other two terms. Using (77) and (79) and integration by parts in momentum space we arrive at

$$\begin{aligned}\delta\Gamma = &- \int \mathrm{d}x\, \partial_\nu w^{\mu\nu} x_\mu (\phi \bar{*} \phi \bar{*} \phi) \\ &- w^{\mu\nu} \int \mathrm{d}\mu(p_i)\ \sum_i p_{i\mu} \frac{\partial}{\partial p_{i\nu}} F(\overline{p}_1, \overline{p}_2, \overline{p}_3) \\ &+ \int \mathrm{d}x \partial_\rho w^{\mu\nu} \int \mathrm{e}^{-\mathrm{i}x(p_1+p_2+p_3)} \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3) \\ &\qquad \times F(\overline{p}_1, \overline{p}_2, \overline{p}_3) \\ &\qquad \times \sum_i S^\rho_i \Phi_i p_{i\mu} \frac{\partial}{\partial p_{i\nu}} (\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)).\end{aligned} \tag{92}$$

Now the first and third terms vanish for a global Lorentz transformation (for which $w^{\mu\nu}$ is constant). The second term represents an infinitesimal Lorentz transformation in momentum space of $F(\overline{p}_1, \overline{p}_2, \overline{p}_3)$. Note that

$$\delta F(\overline{p}_1, \overline{p}_2, \overline{p}_3) := \sum_i w^\mu{}_\nu p_{i\mu} \frac{\partial}{\partial p_{i\nu}} F(\overline{p}_1, \overline{p}_2, \overline{p}_3) \tag{93}$$

$$= F(\overline{p}_1, \overline{p}_2, \overline{p}_3) \times \sum_i w^\mu{}_\nu p_{i\mu} \frac{\partial}{\partial p_{i\nu}} (\overline{p}_1 \wedge \overline{p}_2 + \overline{p}_2 \wedge \overline{p}_3 + \overline{p}_1 \wedge \overline{p}_3). \tag{94}$$

Now for an infinitesimal $SO(1,1) \times SO(2)$ transformation $w^u{}_u = -w^v{}_v$ and $w^2_3 = -w^3_2$ are the only non-zero components of w^μ_ν and one can easily show that

$$w^\mu{}_\nu p_\mu \partial^\nu \overline{p}_\rho = \overline{p}_\mu w^\mu{}_\rho, \quad w \in so(1,1) \times so(2), \tag{95}$$

which is the statement that $\overline{p}$ is covariant under $SO(1,1) \times SO(2)$. Note that this is not true for an arbitrary infinitesimal Lorentz transformation. In particular there are extra non-covariant terms in the above equation involving $w^u{}_a$.

It is now easy to see that $\delta F = 0$ under an $SO(1,1) \times SO(2)$ transformation since we know that $\theta^{\mu\nu}$ is invariant (i.e. $w^\mu{}_{\mu'}\theta^{\mu'\nu} + \theta^{\mu\nu'}w^\nu{}_{\nu'} = 0$).

Having thus proven invariance of the tree-level quantum theory under $SO(1,1) \times SO(2)$ we can then construct the angular-momentum tensor simply by reading off the coefficient of $\partial_\rho w^{\mu\nu}$ in (92). We obtain

$$\begin{aligned}M^{\rho\mu\nu} = &\eta^{\rho[\nu} x^{\mu]} \Gamma_{\mathrm{int}} \\ &+ \int \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3 \mathrm{e}^{-\mathrm{i}x(p_1+p_2+p_3)} \\ &\quad \times F(\overline{p}_1, \overline{p}_2, \overline{p}_3) \\ &\quad \times \sum_i S^\rho_i \Phi_i p_i^{[\mu} \partial^{\nu]} \left(\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\right).\end{aligned} \tag{96}$$

6 LSZ reduction

In ordinary quantum field theory one obtains matrix elements of operators from time ordered Green functions by the LSZ reduction formulae. Since they are based on the usual time ordering with respect to x^0 we have to check that also in our case we have analogous relations.

The quantity most immediately associated to Green functions of interacting fields is the S-matrix. One can straightforwardly mimic the manipulations used to derive the standard LSZ reduction formulae if we postulate the existence of an asymptotic (weak) limit

$$\sqrt{z}\Phi_{\mathrm{in}}(x) = \lim_{u \to -\infty} \Phi(x), \qquad \sqrt{z}\Phi_{\mathrm{out}}(x) = \lim_{u \to +\infty} \Phi(x). \tag{97}$$

Here the factor z corresponds to the wave function renormalisation and will be suppressed in the formulae to follow. The result is that an arbitrary matrix element with n out- and l incoming particles is related to Green functions of interacting fields as follows:

$$_{\mathrm{out}}\langle p_1 \cdots p_n | q_1 \cdots q_l \rangle_{\mathrm{in}} = {}_{\mathrm{in}}\langle p_1 \cdots p_n | S | q_1 \cdots p_l \rangle_{\mathrm{in}} \tag{98}$$

$$= \mathrm{i}^{n+l} \int \mathrm{d}^4 y_1 \cdots \mathrm{d}^4 x_l \mathrm{e}^{\mathrm{i}p_k y_k + q_j x_j} \times (\Box_{y_1} + m^2) \cdots (\Box_{x_l} + m^2) \times \langle 0 | T\Phi(y_1) \cdots \Phi(x_l) | 0 \rangle \tag{99}$$

$$= (-\mathrm{i})^{n+1} (p_1^2 - m^2) \cdots (q_l^2 - m^2) \tilde{G}(p_1, \cdots, q_l)|, \tag{100}$$

where $\tilde{G}$ is the Fourier transform of the time ordered Green's function and the vertical line indicates that all momenta are put on-shell by setting $p_u = (p_a p_a + m^2)/(2p_v)$, S is the S-matrix and the time ordering is with respect to the u component.

7 Comparison of Feynman rules for different formulations of non-commutative field theories

7.1 The naïve Feynman rules

The naïve Feynman rules correspond to taking the Gell-Mann–Low formula but assuming that all derivatives in the interaction Lagrangian occur outside the time ordering. We sketch the naïve momentum space Feynman rules (up to factors) for ϕ^3_* theory looking at a diagram with N internal lines, with momenta k_i, E external lines with

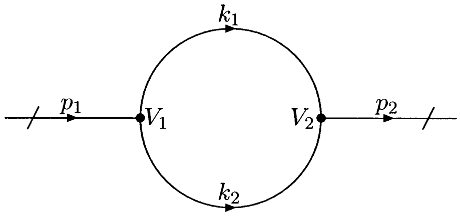

Fig. 1. 1-loop diagram. The momenta r_j, s_j, t_j can be read off from the diagram: $(r_1, s_1, t_1) = (p_1, -k_1, -k_2)$ $(r_2, s_2, t_2) = (k_1, k_2, -p_2)$

momentum p_i and V vertices. The jth vertex has lines entering it with momenta r_j, s_j, t_j which could be internal or external lines. The Feynman rules for this diagram as given for example by Filk [15] result in

$$\begin{aligned}\tilde{G}(p_i) \\ \sim S^{-1}\prod_{i=1}^{E} P(p_i)\prod_{i=1}^{N}\int \mathrm{d}^4k_i P(k_i) \times \prod_{j=1}^{V}\delta^4(r_j+s_j+t_j) \\ \times F(r_j, s_j, t_j),\end{aligned} \tag{101}$$

where S is a symmetry factor, $P(k_i) = \frac{\mathrm{i}}{k_i^2-m^2+\mathrm{i}\epsilon}$ is the Fourier transform of the propagator and $F(p_1, p_2, p_3)$ is the non-commutative phase factor at the vertex given by ϕ_*^3 theory. We have

$$F(p_1,p_2,p_3) = \sum_{\sigma\in P_3} \mathrm{e}^{\mathrm{i}(p_{\sigma(1)}\wedge p_{\sigma(2)}+p_{\sigma(1)}\wedge p_{\sigma(3)}+p_{\sigma(2)}\wedge p_{\sigma(3)})}, \tag{102}$$

where P_3 is the set of permutations of $(1, 2, 3)$. The momenta r_j, s_j, t_j are the momenta entering a vertex V_j and can be read off from the Feynman diagram. We illustrate this with a 1-loop two-point function in Fig. 1.

So the Green's function corresponding to this diagram is

$$\begin{aligned}\tilde{G}(p_1,p_2) = S^{-1}P(p_1)P(p_2)\int \mathrm{d}^4k_1\mathrm{d}^4k_2 P(k_1)P(k_2) \\ \times\delta^4(p_1-k_1-k_2)\delta^4(k_1+k_2-p_2) \\ \times F(p_1,-k_1,-k_2)F(k_1,k_2,-p_2).\end{aligned} \tag{103}$$

It is by now well known that these rules respect unitarity only in the case that θ_e vanishes since otherwise there is a conflict of commuting time derivatives of the star product with time ordering.

7.2 TOPT time ordered with respect to x^0

In [2] Feynman rules were also derived by following the Gell-Mann–Low formula using the usual (i.e. with respect to x^0) time ordering but proper care had been taken to the occurrence of time derivatives from the star product before time ordering. These rules also follow from the Hamiltonian approach of [1]. We now associate a number $\lambda_i = \pm 1$ with each internal momentum k_i and a number $\mu_i = \pm 1$ with each external momentum p_i. The resulting Feynman rules are

$$\begin{aligned}\tilde{G}(p_i) \sim S^{-1}\sum_{\lambda_i,\mu_i}\prod_{i=1}^{E} P_{\mu_i}(p_i)\prod_{i=1}^{N}\int \mathrm{d}^4k_i P_{\lambda_i}(k_i) \\ \times\prod_{j=1}^{V}\delta^4(r_j+s_j+t_j)F(r_j^\lambda, s_j^\lambda, t_j^\lambda).\end{aligned} \tag{104}$$

Here the momenta appearing in the phase factor are put on-shell by replacing the zeroth component of p, with λw_p as indicated by the superscript λ. The notation here is somewhat schematic: the superscript λ is that associated with the momentum r_j, s_j or t_j. We must then sum over $\lambda_i = \pm 1$ corresponding to positive and negative frequency momenta. The factor

$$P_\lambda(k) = \frac{\lambda}{2w_k(k^0-\lambda(w_k-\mathrm{i}\epsilon))} = \frac{\eta_\lambda(k)}{k^2-m^2+\mathrm{i}\epsilon}, \tag{105}$$

$$\eta_\lambda(k) = 1/2(1+\lambda k_0/w_k), \tag{106}$$

$$w_k = \sqrt{\boldsymbol{k}^2+m^2} \tag{107}$$

is the Fourier transform of $\theta(\lambda x^0)D^\lambda(x)$. Note that for on-shell momenta this is equal to the propagator whereas even for off-shell momenta we have that $P_+(k)+P_-(k) = P(k)$. This implies that if the non-commutative phase factor is independent of λ (as for example in the case of pure space-space non-commutativity) then summing over λ we obtain the naïve Feynman rules (101). Since, however, the phase factor does explicitly depend on λ in the generic case (i.e. with $\theta_e \neq 0$) we cannot re-express these rules in terms of ordinary propagators.

For the 1-loop diagram above we obtain

$$\begin{aligned}\tilde{G}(p_1,p_2) = S^{-1}\sum_{\lambda_i,\mu_i} P_{\mu_1}(p_1)P_{\mu_2}(p_2) \\ \times\int \mathrm{d}^4k_1\mathrm{d}^4k_2 P_{\lambda_1}(k_1)P_{\lambda_2}(k_2) \\ \times\delta(p_1-k_1-k_2)\delta(k_1+k_2-p_2) \\ \times F(p_{1\mu_1}, -k_{1\lambda_1}, -k_{2\lambda_2})F(k_{1\lambda_1}, k_{2\lambda_2}, -p_{2\mu_2}).\end{aligned} \tag{108}$$

These rules lead to unitary amplitudes also in the case when θ_e does not vanish, but still they are not satisfactory: the main drawback being that the underlying $SO(1,1)\times SO(2)$ invariance is not maintained.

7.3 TOPT with the new time ordering

Finally we consider the Feynman rules obtained from the Gell-Mann–Low formula with the new time ordering introduced in Sect. 3.3. We could derive the Feynman rules from first principles, but in order to compare this approach with the previous one, we instead derive the new Feynman rules by suitably adapting (104). We only need to find modifications for $P_\lambda(k)$ and for the non-commutative phase factor. Since $P_\lambda(k)$ is the Fourier transform of $\theta(\lambda x^0)D^\lambda(x)$ we replace this with the Fourier transform of $\theta(\lambda u)D^\lambda(x)$. Using

$$\theta(\lambda u) = \frac{\mathrm{i}\lambda}{2\pi}\int \frac{\mathrm{d}s\mathrm{e}^{-\mathrm{i}su}}{s+\mathrm{i}\epsilon\lambda}, \quad (109)$$

$$D^\lambda(x) = \int \frac{\mathrm{d}^3 p}{2p_v}\theta(\lambda p_v)\mathrm{e}^{-\mathrm{i}\overline{p}x}, \quad (110)$$

we obtain

$$P_\lambda(k) \to \frac{\theta(\lambda k_v)}{2k_v(k_u - \bar{k}_u + \mathrm{i}\epsilon\lambda)} = \frac{\theta(\lambda k_v)}{k^2 - m^2 + \mathrm{i}\epsilon}. \quad (111)$$

The non-commutative phase factor is obtained by taking the naïve phase factor F, putting all the momenta on-shell, and splitting into positive and negative frequency parts. In the present case we put the momenta on-shell by replacing p with $\overline{p}$ (see (25)) and the positive and negative frequency parts correspond to positive and negative p_v. So we expect the non-commutative phase factor

$$\begin{aligned} &F(k_1^{\lambda_1}, k_2^{\lambda_2}, k_3^{\lambda_3}) \\ &\to \theta(\lambda_1 k_{1v})\theta(\lambda_2 k_{2v})\theta(\lambda_3 k_{3v})F(\overline{k}_1, \overline{k}_2, \overline{k}_3). \end{aligned} \quad (112)$$

Note that in this case splitting into positive and negative frequency parts simply corresponds to taking p_v positive or negative (which we have indicated by using step functions). However the step functions are already present in $P_\lambda(x)$ so the phase factor is effectively independent of λ. Indeed, if we perform the sum over λ the P_λs sum to give complete propagators and we obtain the following Feynman rules:

$$\begin{aligned} &\tilde{G}(p_i) \\ &\sim S^{-1}\prod_{i=1}^{E} P(p_i)\prod_{i=1}^{N}\int \mathrm{d}k_i P(k_i) \times \prod_{j=1}^{V}\delta(r_j + s_j + t_j) \\ &\times F(\overline{r}_j, \overline{t}_j, \overline{s}_j). \end{aligned} \quad (113)$$

Note that the only difference to the naïve case is the appearance of the modified phase factor.

So the Green's function corresponding to the 1-loop diagram above is

$$\begin{aligned} &\tilde{G}(p_1, p_2) \\ &= S^{-1}P(p_1)P(p_2)\int \mathrm{d}^4k_1\mathrm{d}^4k_2 P(k_1)P(k_2) \\ &\quad\times\delta^4(p_1 - k_1 - k_2)\delta^4(k_1 + k_2 - p_2) \\ &\quad\times F(\overline{p}_1, -\overline{k}_1, -\overline{k}_2)F(\overline{k}_1, \overline{k}_2, -\overline{p}_2). \end{aligned} \quad (114)$$

Clearly this is a tremendous simplification when actually calculating diagrams. In a sense the new time ordering rendered superfluous the explicit distinction between positive and negative frequency parts and thus reunited what had to be separated in old fashioned time ordered perturbation theory. It also seems to obey the positive energy condition discussed in [16] since the free propagator certainly does and in diagrams describing interaction also energy components occur with the correct signs only. This is ensured by the fact that we can formulate Feynman rules in terms of propagators.

Of course, as for TOPT [3] and the equivalent Hamiltonian formulation [1] unitarity is now automatic due to a correct treatment of the time ordering. In the appendix we show this explicitly by checking the optical theorem.

8 Discussion, conclusions and outlook

The starting point for the considerations of the present paper is the symmetry content of a theory of quantum fields if one has interactions according to the Moyal product and a generic $\theta_{\mu\nu}$. It comprises translations and $SO(1,1)\times SO(2)$ which should be maintained in the course of quantization. Basing time ordering on the values x^0 of the coordinates this is not the case. It is however true when we time order according to the light-wedge variable $u = (x^0 - x^1)/\sqrt{2}$. We proved the symmetry content of the theory to be the desired one by explicitly constructing the respective conserved currents in the form of Ward identities for time ordered Green functions. Here we used the gratifying fact that the quantum equations of motion can be written in closed form via an effective action. It is remarkable that this effective action is in the tree approximation not the classical action but the classical action with star product modified into products living on the mass-shell of the fields.

A further noticeable simplification arises on the level of Feynman rules. Again as a consequence of the new time ordering we arrive at essentially naïve ones: propagators are the usual ones, phase factors are those of the modified star product, i.e. mass-shell factors written in the new variables. Note that one might be worried about infrared divergences arising from using light-cone coordinates (as for example $p_v \to 0$). These Feynman rules show that these are unlikely to occur since the only divergent pieces occur in the phase where they are rendered harmless.

Unitarity has been checked to hold in an explicit example which however permits immediate generalisation on a formal level. Hence the theory is certainly well defined on the tree level.

LSZ reduction works well when the limit of u going to plus or minus infinity is taken as defining the asymptotics. Causality is lost in the sense that there are in general no two points x, y in space-time where we can be certain that $\Phi(x)$ commutes with $\Phi(y)$ (i.e. no analogue of "space-like separation"). This means that our time ordering defines a genuine "before" and "after". There is no ambiguous (space-like) region as there is in an ordinary relativistic quantum field theory.

Thinking of extensions of our results one can indeed have the hope that gauge theories exist as well for generic θ, since global symmetry currents will exist due to the simple form of the quantum equations of motion. Hence the examples which are known to exist for vanishing θ_e should all be generalisable to generic θ. In the actual formulation of gauge fixing and BRS invariance the expertise collected in light-cone quantization should be helpful. For higher orders analogously one should at least be able to construct

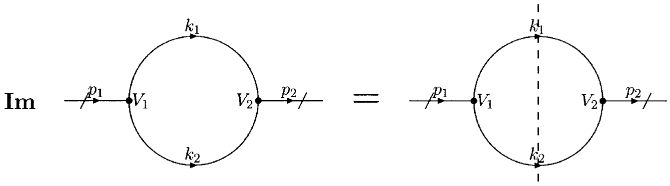

Fig. 2. Optical theorem for the 1-loop diagram. The imaginary part of the amputated 1-loop diagram equals the cut graph on the right which will be defined below

what can be constructed for restricted θ. Since with the time ordering the integrals truly change one should also have a fresh look at the ultraviolet/infrared connection. It may very well differ from the previous one.

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Appendix: Unitarity

Having derived the Feynman rules we are now in a position to test unitarity of the theory by checking the optical theorem. As an example we study the two-point function of ϕ^3_* in the one-loop approximation. The optical theorem can be given diagrammatically as Fig. 2.

The proof of unitarity follows closely that of [2] for the usual time ordering. The left-hand side of this equation is the imaginary part of the 1-loop function given in (114), amputated by removing the terms $P(p_1)P(p_2)$. In order to find the imaginary part of this we first explicitly perform the integration over the uth component of all internal momenta $(k_i)_u$. For this it is crucial that unlike the naïve Feynman rules, here the non-commutative phase factor F is independent of $(k_i)_u$. The integration over $(k_1)_u$ can be finished by the delta function and the integration over k_2 can be performed using contour integration. The result is

$$\begin{aligned} &\frac{\tilde{G}(p_1,p_2)}{P(p_1)P(p_2)} \\ &= S^{-1}\int \mathrm{d}^3\mu_1 \mathrm{d}^3\mu_2 \frac{\delta^3(p_1-k_1-k_2)\delta^4(p_1-p_2)}{p_{1u}-\overline{k}_{1u}-\overline{k}_{2u}+\mathrm{i}\epsilon} \\ &\quad \times F(\overline{p}_1,-\overline{k}_1,-\overline{k}_2)F(\overline{k}_1,\overline{k}_2,-\overline{p}_2), \end{aligned} \tag{A.1}$$

where $\mathrm{d}^3\mu_i = \mathrm{d}k_v \mathrm{d}k_2 \mathrm{d}k_3/2k_v$ is the invariant measure.

We are now in a position to compute the imaginary part of this. Using the distribution identity

$$\mathrm{Im}\left(\frac{1}{x+\mathrm{i}\epsilon}\right) = \delta(x+\mathrm{i}\epsilon), \tag{A.2}$$

we find the left-hand side of the optical theorem:

$$\begin{aligned} &\mathrm{Im}\frac{\tilde{G}(p_1,p_2)}{P(p_1)P(p_2)} \\ &= S^{-1}\int \mathrm{d}^3\mu_1 \mathrm{d}^3\mu_2 \delta^4(p_1-\overline{k}_1-\overline{k}_2)\delta^4(p_1-p_2) \\ &\quad \times F(\overline{p}_1,-\overline{k}_1,-\overline{k}_2)F(\overline{k}_1,\overline{k}_2,-\overline{p}_2) \\ &= S^{-1}\int \mathrm{d}^4k_1 \mathrm{d}^4k_2 \delta^4(p_1-k_1-k_2)\delta^4(k_1+k_2-p_2) \\ &\quad \times \delta^4(k_1^2-m^2)\delta^4(k_2^2-m^2) \\ &\quad \times F(\overline{p}_1,-\overline{k}_1,-\overline{k}_2)F(\overline{k}_1,\overline{k}_2,-\overline{p}_2). \end{aligned} \tag{A.3}$$

Comparing with (114) we see that the imaginary part of the amputated Green's function is obtained simply by replacing the propagators with delta functions, $P(k) \to \delta(k^2-m^2)$.

The right-hand side of the optical theorem is defined to be

$$\begin{aligned} &\int \mathrm{d}^4k_1 \mathrm{d}^4k_2 \delta(k_1^2-m^2)\delta(k_2^2-m^2) \\ &\quad \times \mathcal{M}^*(-p_2 \to k_1k_2)\mathcal{M}(p_1 \to k_1k_2) \end{aligned} \tag{A.4}$$

and by substituting in

$$\mathcal{M}(p_1 \to k_1k_2) = F(\overline{p}_1,-k_1,-k_2)\delta^4(p_1-k_1-k_2), \tag{A.5}$$

$$\mathcal{M}^*(p_2 \to k_1k_2) = F(\overline{p}_1,-k_1,-k_2)\delta^4(p_1-k_1-k_2), \tag{A.6}$$

we find that the right-hand side equals the left-hand side and the optical theorem is satisfied.

As usual, in this unitarity check one had to be sure only of the fact that the imaginary part of the loop diagram is finite. The real part diverges and would require proper definition which we do not attempt here. On this formal level one can also state generalisations: unitarity will be alright to all orders with our time ordering since our non-commutative phase factors do not change the unitarity character of an underlying unitary theory.

References

1. S. Doplicher, K. Fredenhagen, J.E. Roberts, Phys. Lett. B **331**, 39 (1994); Commun. Math. Phys. **172**, 187 (1995) [hep-th/0303037]
2. Y. Liao, K. Sibold, Eur. Phys. J. C **25**, 469 (2002)
3. Y. Liao, K. Sibold, Eur. Phys. J. C **25**, 479 (2002)
4. C. Rim, J.H. Yee, Phys. Lett. B **574**, 111 (2003)
5. D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli, Phys. Lett. B **533**, 178 (2002)
6. Y. Liao, unpublished
7. T. Ohl, R. Rückl, J. Zeiner, Nucl. Phys. B **676**, 229 (2004)
8. T. Reichenbach, hep-th/0411127; Leipzig Universtiy Diploma thesis (2004)
9. O. Aharony, J. Gomis, T. Mehen, JHEP **09**, 023 (2000)
10. J. Simon, JHEP **06**, 001 (2002)
11. H. Epstein, N. Cimento **27**, 886 (1963)
12. B. Schroer, unpublished preprint (1963)
13. L. Alvarez-Gaume, M.A. Vazquez-Mozo, Nucl. Phys. B **668**, 293 (2003)
14. P. Heslop, K. Sibold, to appear
15. T. Filk, Phys. Lett. B **376**, 53 (1996)
16. K. Fujikawa, hep-th/0410146